Common Fixed Point Theorems in $\mathcal{M}$-Fuzzy Cone Metric Spaces

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Abstract
This work aims to generalize the Banach contraction theorem to $\mathcal{M}$-fuzzy cone metric spaces. We construct generalized $\mathcal{M}$-fuzzy cone contractive conditions for three self mappings with which they have a unique common fixed point.

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1. Introduction

Fuzzy sets that handle uncertainties well was introduced by Zadeh [10]. Huang and Zhang [4] introduced cone and defined cone metric spaces as a generalization of metric spaces [1]. Tarkan Oner et al. [9] introduced fuzzy cone metric spaces that generalized fuzzy metric spaces [2]. These ideas motivated the researchers to come up with several new ideas as they act as a base for introducing new concepts and proving many more new results. The aim here is to construct and prove $\mathcal{M}$-Fuzzy Cone Banach Contraction Theorem and some common fixed point theorems for three self mappings which satisfy generalized contractive conditions in $\mathcal{M}$-Fuzzy Cone Metric Spaces and to provide an example to exhibit the same.

2. Preliminaries

Definition 1. [4] Let $\mathcal{B}$ be a real Banach space and $\mathcal{C}$ be a subset of $\mathcal{B}$. $\mathcal{C}$ is called a cone if and only if:

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[C1] \( C \) is nonempty, closed and \( C \neq \{0\} \),

[C2] \( \rho, \sigma \in \mathbb{R}, \rho, \sigma \geq 0, c_1, c_2 \in C \) imply \( \rho c_1 + \sigma c_2 \in C \),

[C3] \( c \in C \) and \( -c \in C \) imply \( c = 0 \).

The cones considered here are subsets of a real Banach space and are with nonempty interiors.

**Definition 2.** An \( \mathcal{M} \)-Fuzzy Cone Metric Space (briefly, \( \mathcal{M} \)-FCM Space) is a 3-tuple \((Z, \mathcal{M}, *)\) where \( Z \) is an arbitrary set, * is a continuous \( t \)-norm, \( C \) is a cone and \( \mathcal{M} \) a fuzzy set in \( \mathcal{Z}^3 \times \text{int}(C) \) satisfying the following conditions: For all \( \zeta, \eta, \omega, u \in Z \) and \( c, c' \in \text{int}(C) \),

\[
\begin{align*}
\text{MFC1} & : \mathcal{M}(\zeta, \eta, \omega, c) > 0, \\
\text{MFC2} & : \mathcal{M}(\zeta, \eta, \omega, c) = 1 \text{ if and only if } \zeta = \eta = \omega, \\
\text{MFC3} & : \mathcal{M}(\zeta, \eta, \omega, c) = \mathcal{M}(p(\zeta, \eta, \omega), c), \text{ where } p \text{ is a permutation}, \\
\text{MFC4} & : \mathcal{M}(\zeta, \eta, \omega, c + c') \geq \mathcal{M}(\zeta, \eta, u, c) \ast \mathcal{M}(u, \omega, c'), \\
\text{MFC5} & : \mathcal{M}(\zeta, \eta, \omega, \cdot) : \text{int}(C) \to [0, 1] \text{ is continuous}. 
\end{align*}
\]

Then \( \mathcal{M} \) is called an \( \mathcal{M} \)-Fuzzy Cone Metric on \( Z \). The function \( \mathcal{M}(\zeta, \eta, \omega, c) \) denotes the degree of nearness between \( \zeta, \eta \) and \( \omega \) with respect to \( c \).

**Example 3.** Let \( \mathcal{B} = \mathbb{R} \) and consider the cone \( C = [0, +\infty) \) in \( \mathcal{B} \). Consider an increasing continuous function \( g : \mathcal{C} \to \mathcal{C} \) and \( a, b > 0 \). Let the \( t \)-norm * be defined by \( \rho * \sigma = \rho \sigma \). Define \( \mathcal{M} : \mathbb{R}^3 \times \text{int}(C) \to [0, 1] \) by

\[
\mathcal{M}(\zeta, \eta, \omega, c) = \left( \frac{\min\{f(x), f(y), f(z)\}^a + \|g(c)\|}{\max\{f(x), f(y), f(z)\}^a + \|g(c)\|} \right)^b
\]

for all \( \zeta, \eta, \omega \in \mathbb{R} \) and \( c \in \text{int}(C) \). Then \((\mathbb{R}, \mathcal{M}, *)\) is an \( \mathcal{M} \)-FCM Space.

**Definition 4.** A symmetric \( \mathcal{M} \)-FCM Space is an \( \mathcal{M} \)-FCM Space \((Z, \mathcal{M}, *)\) satisfying

\[
\mathcal{M}(\eta, \omega, \omega, c) = \mathcal{M}(\omega, \eta, \eta, c), \text{ for all } \eta, \omega \in Z \text{ and } c \in \text{int}(C).
\]

**Remark 5.** An \( \mathcal{M} \)-FCM Space is symmetric.

**Definition 6.** Let \((Z, \mathcal{M}, *)\) be an \( \mathcal{M} \)-FCM Space. A self mapping \( \mathcal{P} : Z \to Z \) is said to be \( \mathcal{M} \)-Fuzzy Cone Contractive (briefly, \( \mathcal{M} \)-FCC) if there exists \( k \in (0, 1) \) such that

\[
\left( \frac{1}{\mathcal{M}(\mathcal{P}(\zeta), \mathcal{P}(\eta), \mathcal{P}(\omega), c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta, \eta, \omega, c)} - 1 \right),
\]

for all \( \zeta, \eta, \omega \in Z \) and \( c \in \text{int}(C) \).

**Definition 7.** In an \( \mathcal{M} \)-FCM Space \((Z, \mathcal{M}, *)\), \( \mathcal{M} \) is said to be triangular if, for all \( \zeta, \eta, \omega, u \in Z \) and \( c \in \text{int}(C) \),

\[
\left( \frac{1}{\mathcal{M}(\zeta, \eta, \omega, c)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(\zeta, \eta, u, c)} - 1 \right) + \left( \frac{1}{\mathcal{M}(u, \omega, \omega, c)} - 1 \right).
\]

**Definition 8.** Let \((Z, \mathcal{M}, *)\) be an \( \mathcal{M} \)-FCM Space, \( \zeta' \in Z \) and \( \{\zeta_n\} \) be a sequence in \( Z \).

(i) \( \{\zeta_n\} \) is said to converge to \( \zeta' \) if for all \( c \in \text{int}(\mathcal{C}) \), \( \lim_{n \to +\infty} \left( \frac{1}{\mathcal{M}(\zeta_n, \zeta', \omega, c)} - 1 \right) = 0 \). It is denoted by \( \lim_{n \to +\infty} \zeta_n = \zeta' \) or by \( \zeta_n \to \zeta' \) as \( n \to +\infty \).
(ii) $\{\zeta_n\}$ is said to be a Cauchy sequence if \(\lim_{n \to +\infty} \left( \frac{1}{\mathcal{M}(\zeta_{n+m}, \zeta_n, c)} - 1 \right) = 0\), for all $c \in \text{int}(\mathcal{C})$ and $m \in \mathbb{N}$.

(iii) $(\mathcal{Z}, \mathcal{M}, \ast)$ is called a complete $\mathcal{M}$-FCM space if every Cauchy sequence in $\mathcal{Z}$ converges.

**Definition 9.** Let $(\mathcal{Z}, \mathcal{M}, \ast)$ be an $\mathcal{M}$-FCM Space. A sequence $\{\zeta_n\}$ in $\mathcal{Z}$ is $\mathcal{M}$-Fuzzy Cone Contractive if there exists $k \in (0, 1)$ such that

$$\left( \frac{1}{\mathcal{M}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta_{n-1}, \zeta_n, c)} - 1 \right),$$

for all $c \in \text{int}(\mathcal{C})$.

3. Main Results

Let us first state and prove the $\mathcal{M}$-fuzzy cone Banach contraction theorem in a complete $\mathcal{M}$-FCM Space.

**Theorem 1.** Let $(\mathcal{Z}, \mathcal{M}, \ast)$ be a complete $\mathcal{M}$-FCM Space in which $\mathcal{M}$-FCC sequences are Cauchy. Let $\mathcal{P} : \mathcal{Z} \to \mathcal{Z}$ be an $\mathcal{M}$-FCC mapping. Then $\mathcal{P}$ has a unique fixed point.

**Proof.** Let $\zeta_0 \in \mathcal{Z}$ and $c \in \text{int}(\mathcal{C})$. Define a sequence $\{\zeta_n\}$ by

$$\zeta_n = \mathcal{P}^n \zeta_0, \ n \in \mathbb{N}.$$

Since $\mathcal{P}$ is $\mathcal{M}$-FCC, we have

$$\left( \frac{1}{\mathcal{M}(\mathcal{P} \zeta, \mathcal{P}^2 \zeta, \mathcal{P}^2 \zeta, c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta, \mathcal{P} \zeta, \mathcal{P} \zeta, c)} - 1 \right),$$

for all $\zeta \in \mathcal{Z}$ and for some $k \in (0, 1)$. This gives

$$\left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right).$$

This makes $\{\zeta_n\}$ an $\mathcal{M}$-FCC sequence and by assumption $\zeta_n \to \zeta$ for some $\zeta \in \mathcal{Z}$.

Now,

$$\left( \frac{1}{\mathcal{M}(\mathcal{P} \zeta_n, \mathcal{P} \zeta, \mathcal{P} \zeta, c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta_n, \zeta, c)} - 1 \right).$$

As $k < 1$,

$$\lim_{n \to +\infty} \left( \frac{1}{\mathcal{M}(\mathcal{P} \zeta_n, \mathcal{P} \zeta, \mathcal{P} \zeta, c)} - 1 \right) = 0.$$

That is,

$$\left( \frac{1}{\mathcal{M}(\zeta, \mathcal{P} \zeta, \mathcal{P} \zeta, c)} - 1 \right) = 0,$$

and which gives $\mathcal{P} \zeta = \zeta$. 


Suppose $P\eta = \eta$, for some $\eta \in \mathbb{Z}$. Then
\[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, \eta, c)} - 1 \right) = \left( \frac{1}{\mathcal{M}(P\zeta, P\zeta, P\eta, c)} - 1 \right) \\
\leq k \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \eta, c)} - 1 \right) \\
= \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \eta, c)} - 1 \right) \\
\leq k^2 \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \eta, c)} - 1 \right) \\
\cdots \\
\leq k^n \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \eta, c)} - 1 \right) \\
\rightarrow 0 \text{ as } n \rightarrow +\infty.
\]
Therefore $\zeta = \eta$.

The following theorem considers three self mappings and proves the existence of their unique fixed point under a generalized contractive condition in a complete $\mathcal{M}$-FCM Space.

**Theorem 2.** Let $(\mathbb{Z}, \mathcal{M}, \ast)$ be a complete $\mathcal{M}$-FCM Space where $\mathcal{M}$ is triangular. If $P, Q, R : \mathbb{Z} \rightarrow \mathbb{Z}$ is such that for all $\zeta, \eta, \omega \in \mathbb{Z}$ and $c \in \text{int}(\mathbb{C})$,
\[
\left( \frac{1}{\mathcal{M}(P\zeta, Q\eta, R\omega, c)} - 1 \right) \leq \left\{ k_1 \left( \frac{1}{\mathcal{M}(\zeta, \eta, \omega, c)} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\eta, \omega, c)} - 1 \right) + k_3 \left( \frac{1}{\mathcal{M}(\omega, c)} - 1 \right) + k_4 \left( \frac{1}{\mathcal{M}(\zeta, c)} - 1 \right) \right\} (2.1)
\]
where $k_i \in [0, +\infty], i = 1, \ldots, 4$ and $k_1 + 2(k_2 + k_3) + k_4 < 1$. Then $P, Q$ and $R$ have a unique common fixed point.

**Proof.** Let $\zeta_0 \in \mathbb{Z}$ be arbitrary. Let the sequence $\{\zeta_n\}$ be defined by
\[
\begin{align*}
\zeta_{n+1} & = P\zeta_n, \\
\zeta_{n+2} & = Q\zeta_{n+1}, \text{ and,} \\
\zeta_{n+3} & = R\zeta_{n+2} \quad \text{for } n \geq 0.
\end{align*}
\]
From (2.1),
\[
\begin{align*}
\left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) & \leq \left( \frac{1}{\mathcal{M}(P\zeta_n, Q\zeta_{n+1}, Q\zeta_{n+1}, c)} - 1 \right) \\
& \leq \left\{ k_1 \left( \frac{1}{\mathcal{M}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) \\
& \quad + k_3 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) + k_4 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) \right\} \\
& = \left\{ k_1 \left( \frac{1}{\mathcal{M}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) \\
& \quad + k_3 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) + k_4 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) \right\} \\
& = \left\{ k_1 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right) - 1 \right\}
\end{align*}
\]
Again, using (2.1),

\[
\frac{1}{\mathcal{M}(\zeta_{m+1}, \zeta_{m+2}, c)} - 1 \leq k_1 \left( \frac{1}{\mathcal{M}(\zeta_{m+1}, \zeta_{m+2}, c)} - 1 \right)
\]

\[
+ k_2 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, c)} - 1 \right)
\]

\[
+ k_3 \left( \frac{1}{\mathcal{M}(\zeta_{n+1} \zeta_{n+2}, c)} - 1 \right)
\]

\[
= \left\{ (k_1 + k_2 + k_3) \left( \frac{1}{\mathcal{M}(\zeta_{m+1}, \zeta_{m+2}, c)} - 1 \right) \right\}.
\]

Therefore,

\[
\left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, c)} - 1 \right) \leq \frac{k_1 + k_2 + k_3}{1 - (k_2 + k_3)} \left( \frac{1}{\mathcal{M}(\zeta_{m+1}, \zeta_{m+2}, c)} - 1 \right).
\]

(2.2)

Again, from (2.1),

\[
\frac{1}{\mathcal{M}(\zeta_{m+2}, \zeta_{m+3}, \zeta_{m+3})} - 1 \leq \frac{1}{\mathcal{M}(\Omega_{m+1}, R\zeta_{m+2}, R\zeta_{m+3}) - 1}
\]

\[
\leq k_1 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2})} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2})} - 1 \right)
\]

\[
+ k_3 \left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2})} - 1 \right)
\]

\[
= (k_1 + k_2 + k_3) \left( \frac{1}{\mathcal{M}(\zeta_{m+1}, \zeta_{m+2}, \zeta_{m+2})} - 1 \right)
\]

\[
+ (k_2 + k_3) \left( \frac{1}{\mathcal{M}(\zeta_{m+1}, \zeta_{m+2}, \zeta_{m+2})} - 1 \right)
\]

This gives,

\[
\left( \frac{1}{\mathcal{M}(\zeta_{m+2}, \zeta_{m+3}, \zeta_{m+3})} - 1 \right) \leq \frac{k_1 + k_2 + k_3}{1 - (k_2 + k_3)} \left( \frac{1}{\mathcal{M}(\zeta_{m+1}, \zeta_{m+2}, \zeta_{m+2})} - 1 \right).
\]

(2.3)

Again, using (2.1),

\[
\frac{1}{\mathcal{M}(\zeta_{m+3}, \zeta_{m+3}, \zeta_{m+3})} - 1 \leq \frac{1}{\mathcal{M}(\Omega_{m+2}, \Omega_{m+2}, \Omega_{m+2}) - 1}
\]

\[
\leq k_1 \left( \frac{1}{\mathcal{M}(\zeta_{m+3}, \zeta_{m+4}, \zeta_{m+4})} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta_{m+3}, \zeta_{m+4}, \zeta_{m+4})} - 1 \right)
\]

\[
+ k_3 \left( \frac{1}{\mathcal{M}(\zeta_{m+3}, \zeta_{m+4}, \zeta_{m+4})} - 1 \right)
\]

\[
= \left\{ (k_1 + k_2 + k_3) \left( \frac{1}{\mathcal{M}(\zeta_{m+3}, \zeta_{m+4}, \zeta_{m+4})} - 1 \right) \right\}.
\]
which makes

These inequalities together gives that

For \( n \)

Put \( M_n = \left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+1})} - 1 \right) \) and \( k = \frac{k_1 + k_2 + k_3}{1 - (k_2 + k_3)} \). Then from (2.2) to (2.4) we have the following inequalities:

For \( n = 0, 1, 2, \ldots \),

These inequalities together gives that

which makes \( \{ \zeta_n \} \) an \( M \)-FCC sequence.

Now, \( M \) is triangular and the space \((Z, M, *)\) is symmetric. Therefore we have,

Thus \( \{ \zeta_n \} \) is Cauchy. As \( Z \) is complete, there exists \( \zeta \in Z \) such that

Since \( M \) is triangular,

\[
\left( \frac{1}{M(\zeta, \zeta, \xi, e)} - 1 \right) \leq \left( \frac{1}{M(\zeta, \zeta, \zeta_{n+2}, e)} - 1 \right) + \left( \frac{1}{M(\zeta_{3n+2}, \xi, \xi, e)} - 1 \right).
\]
From (2.1),
\[
\left(\frac{1}{\mathcal{M}(\zeta_{3n+2}, \mathcal{P}_\zeta, \mathcal{P}_\zeta, c)} - 1\right) \leq \left(\frac{1}{\mathcal{M}(\zeta_{3n+1}, \mathcal{P}_\zeta, \mathcal{P}_\zeta, c)} - 1\right)
\]
\[
\leq \left\{ k_1 \left(\frac{1}{\mathcal{M}(\zeta_{3n+1}, 
\zeta_{3n+2}, c)} - 1\right) + k_2 \left(\frac{1}{\mathcal{M}(\zeta_{3n+1}, \zeta_{3n+2}, c)} - 1\right) \right\}
\]
\[
+ k_3 \left(\frac{1}{\mathcal{M}(\zeta_{3n+1}, \mathcal{P}_\zeta, c)} - 1\right) + k_4 \left(\frac{1}{\mathcal{M}(\zeta_{3n+1}, \mathcal{P}_\zeta, c)} - 1\right)
\]
\[
= \left\{ k_1 \left(\frac{1}{\mathcal{M}(3n+1, \zeta_{3n+2}, c)} - 1\right) + k_2 \left(\frac{1}{\mathcal{M}(3n+1, \zeta_{3n+2}, c)} - 1\right) \right\}
\]
\[
+ k_3 \left(\frac{1}{\mathcal{M}(3n+1, \mathcal{P}_\zeta, c)} - 1\right) + k_4 \left(\frac{1}{\mathcal{M}(3n+1, \mathcal{P}_\zeta, c)} - 1\right)
\]
\[
\rightarrow (k_2 + k_3) \left(\frac{1}{\mathcal{M}(\zeta, \mathcal{P}_\zeta, c)} - 1\right) \text{ as } n \to +\infty.
\]
Therefore,
\[
\lim_{n \to +\infty} \sup \left(\frac{1}{\mathcal{M}(\zeta_{3n+2}, \mathcal{P}_\zeta, \mathcal{P}_\zeta, c)} - 1\right) \leq (k_2 + k_3) \left(\frac{1}{\mathcal{M}(\zeta, \mathcal{P}_\zeta, c)} - 1\right) \quad \text{(2.8)}
\]
From (2.7) and (2.8), we have that
\[
\left(\frac{1}{\mathcal{M}(\zeta, \zeta, \mathcal{P}_\zeta, c)} - 1\right) \leq (k_2 + k_3) \left(\frac{1}{\mathcal{M}(\zeta, \zeta, \mathcal{P}_\zeta, c)} - 1\right).
\]

Since \(k_2 + k_3 < 1\), we have
\[
\left(\frac{1}{\mathcal{M}(\zeta, \zeta, \mathcal{P}_\zeta, c)} - 1\right) = 0, \quad \text{and this gives}
\]
\[
\mathcal{P}_\zeta = \zeta.
\]
Since \(\mathcal{M}\) is triangular,
\[
\left(\frac{1}{\mathcal{M}(\zeta, \zeta, \mathcal{Q}_\zeta, \mathcal{Q}_\zeta, c)} - 1\right) \leq \left(\frac{1}{\mathcal{M}(\zeta, \zeta, \zeta_{3n+3}, c)} - 1\right) + \left(\frac{1}{\mathcal{M}(\zeta_{3n+3}, \mathcal{Q}_\zeta, \mathcal{Q}_\zeta, c)} - 1\right) \quad \text{(2.9)}
\]
From (2.1),
\[
\left(\frac{1}{\mathcal{M}(\zeta_{3n+3}, \mathcal{Q}_\zeta, \mathcal{Q}_\zeta, c)} - 1\right) = \left(\frac{1}{\mathcal{M}(\mathcal{R}_{3n+2}, \mathcal{Q}_\zeta, \mathcal{Q}_\zeta, c)} - 1\right)
\]
\[
\leq \left\{ k_1 \left(\frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, c)} - 1\right) + k_2 \left(\frac{1}{\mathcal{M}(\zeta_{3n+2}, \mathcal{Q}_\zeta, \mathcal{Q}_\zeta, c)} - 1\right) \right\}
\]
\[
+ k_3 \left(\frac{1}{\mathcal{M}(\zeta_{3n+2}, \mathcal{P}_\zeta, c)} - 1\right) + k_4 \left(\frac{1}{\mathcal{M}(\zeta_{3n+2}, \mathcal{P}_\zeta, c)} - 1\right)
\]
\[
= \left\{ k_1 \left(\frac{1}{\mathcal{M}(3n+2, \zeta_{3n+3}, c)} - 1\right) + k_2 \left(\frac{1}{\mathcal{M}(3n+2, \mathcal{Q}_\zeta, \mathcal{Q}_\zeta, c)} - 1\right) \right\}
\]
\[
+ k_3 \left(\frac{1}{\mathcal{M}(3n+2, \mathcal{P}_\zeta, c)} - 1\right) + k_4 \left(\frac{1}{\mathcal{M}(3n+2, \mathcal{P}_\zeta, c)} - 1\right)
\]
\[
\rightarrow (k_2 + k_3) \left(\frac{1}{\mathcal{M}(\zeta, \zeta, \mathcal{Q}_\zeta, c)} - 1\right) \text{ as } n \to +\infty.
\]
Therefore,
\[
\lim_{n \to +\infty} \sup \left(\frac{1}{\mathcal{M}(\zeta_{3n+3}, \mathcal{Q}_\zeta, \mathcal{Q}_\zeta, c)} - 1\right) \leq (k_2 + k_3) \left(\frac{1}{\mathcal{M}(\zeta, \zeta, \mathcal{Q}_\zeta, c)} - 1\right). \quad \text{(2.10)}
\]
From (2.9) and (2.10), we have
\[
\left( \frac{1}{\mathcal{M}(\dot{\zeta}, \dot{\zeta}, Q\dot{\zeta}, c)} - 1 \right) \leq (k_2 + k_3) \left( \frac{1}{\mathcal{M}(\dot{\zeta}, \dot{\zeta}, Q\dot{\zeta}, c)} - 1 \right).
\]
Since \( k_2 + k_3 < 1 \), we have
\[
\left( \frac{1}{\mathcal{M}(\dot{\zeta}, \dot{\zeta}, Q\dot{\zeta}, c)} - 1 \right) = 0, \quad \text{and this gives} \quad Q\dot{\zeta} = \dot{\zeta}.
\]
Since \( \mathcal{M} \) is triangular,
\[
\left( \frac{1}{\mathcal{M}(\dot{\zeta}, \dot{\zeta}, R\dot{\zeta}, c)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(\dot{\zeta}, \dot{\zeta}, \zeta_{3n+1}, c)} - 1 \right) + \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \dot{\zeta}, c)} - 1 \right). \quad (2.11)
\]
From (2.1),
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \dot{\zeta}, c)} - 1 \right) = \left( \frac{1}{\mathcal{M}(P\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c)} - 1 \right) \leq \left\{ \begin{array}{ll}
k_1 \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, \zeta_{3n}, R\dot{\zeta}, c)} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \zeta_{3n}, c)} - 1 \right) \\
+ k_3 \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \zeta_{3n}, c)} - 1 \right) + k_4 \left( \frac{1}{\mathcal{M}(P\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c)} - 1 \right)
\end{array} \right. \\
= \left\{ \begin{array}{ll}
k_1 \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, \zeta_{3n}, R\dot{\zeta}, c)} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \zeta_{3n}, c)} - 1 \right) \\
+ k_3 \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \zeta_{3n}, c)} - 1 \right) + k_4 \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, \zeta_{3n}, \zeta_{3n+1}, c)} - 1 \right)
\end{array} \right. \\
\rightarrow (k_2 + k_3) \left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\dot{\zeta}, c)} - 1 \right) \text{ as } n \rightarrow +\infty.
\]
Therefore,
\[
\lim_{n \rightarrow +\infty} \sup \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \dot{\zeta}, c)} - 1 \right) \leq (k_2 + k_3) \left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\dot{\zeta}, c)} - 1 \right) \quad (2.12)
\]
From (2.11) and (2.12), we have that
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\dot{\zeta}, \dot{\zeta}, c)} - 1 \right) \leq (k_2 + k_3) \left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\dot{\zeta}, c)} - 1 \right).
\]
Since \( k_2 + k_3 < 1 \), we have \( \left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\dot{\zeta}, c)} - 1 \right) = 0 \), and this gives
\[
R\dot{\zeta} = \dot{\zeta}.
\]
Thus we have shown that
\[
P\dot{\zeta} = Q\dot{\zeta} = R\dot{\zeta} = \dot{\zeta}.
\]
Suppose $\mathcal{P}\zeta = \Omega\zeta = \mathcal{R}\zeta = \zeta$. Then from (2.1),
\[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right) = \left( \frac{1}{\mathcal{M}(\mathcal{P}\zeta, \Omega\zeta, \mathcal{R}\zeta, c)} - 1 \right)
\leq \left\{ \begin{array}{l}
k_1 \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right) \\
+k_3 \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right) + k_4 \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right)
\end{array} \right\}
\]
\[
= \left( k_1 + k_2 + k_3 + k_4 \right) \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right).
\]
That is, \[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right) \leq (k_1 + k_2 + k_3 + k_4) \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right).
\]
Therefore, \[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, c)} - 1 \right) = 0, \quad \text{since} \quad k_1 + k_2 + k_3 + k_4 < 1.
\]
Hence we can conclude that $\zeta$ is the unique common fixed point of $\mathcal{P}, \Omega$ and $\mathcal{R}$. 

**Corollary 3.** Let $(\mathcal{M}, \ast)$ be a complete $\mathcal{M}$-FCM Space where $\mathcal{M}$ is triangular. If $\mathcal{P} : X \rightarrow X$ is such that for all $\zeta, \eta, \omega \in X$ and $c \in \text{int}(\mathcal{C})$,
\[
\left( \frac{1}{\mathcal{M}(\mathcal{P}\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c)} - 1 \right) \leq \left\{ \begin{array}{l}
k_1 \left( \frac{1}{\mathcal{M}(\zeta, \eta, \omega, c)} - 1 \right) + k_2 \left( \frac{1}{\mathcal{M}(\zeta, \eta, \omega, c)} - 1 \right) \\
+k_3 \left( \frac{1}{\mathcal{M}(\zeta, \eta, \omega, c)} - 1 \right) + k_4 \left( \frac{1}{\mathcal{M}(\zeta, \eta, \omega, c)} - 1 \right)
\end{array} \right\},
\]
where $k_i \in [0, +\infty], i = 1, \ldots, 4$ and $k_1 + 2(k_2 + k_3) + k_4 < 1$. Then $\mathcal{P}$ has unique fixed point.

**Corollary 4.** Theorem 3 gives Theorem 7 when $\mathcal{P} = \Omega = \mathcal{R}$ and $k_2 = k_3 = k_4 = 0$.

where $\mathcal{C}$ is and a continuous $t$-norm $\ast$.

**Example 5.** Consider $(\mathcal{M}, \ast)$ in which $\mathcal{M} : \mathbb{Z}^3 \times (0, +\infty) \rightarrow [0, 1]$ by
\[
\mathcal{M}(\zeta, \eta, \omega, c) = \frac{|c|}{|c| + (|\zeta - \eta| + |\eta - \omega| + |\omega - \zeta|)} \quad \text{for all} \quad \zeta, \eta, \omega \in \mathcal{Z}\text{ and } c \in \text{int}(\mathcal{C})
\]
where $\mathcal{Z} = \{1, 2, 3\}$ and $\mathcal{C} = \mathbb{R}^+$. Then it is clear that $(\mathcal{M}, \ast)$ is a complete $\mathcal{M}$-FCM Space and that $\mathcal{M}$ is triangular. Consider the self mappings $\mathcal{P}, \Omega$ and $\mathcal{R}$ from $\mathcal{Z}$ to $\mathcal{Z}$, given by $P(1) = 1, P(2) = 2, P(3) = 1, Q(1) = 1, Q(2) = 2, Q(3) = 2, R(1) = 3, R(2) = 2$ and $R(3) = 2$. Then each one of $\mathcal{P}, \Omega$ and $\mathcal{R}$ is not $\mathcal{M}$-FCC and it is not possible for the $\mathcal{M}$-fuzzy cone Banach contraction theorem to assure the existence of their respective fixed points. But $\mathcal{P}, \Omega$ and $\mathcal{R}$ together satisfies the condition (2.1) with $k_1 = \frac{1}{10}, k_2 = \frac{1}{25}, k_3 = \frac{1}{25}$ and $k_4 = \frac{3}{5}$. Therefore $\mathcal{P}, \Omega$ and $\mathcal{R}$ have a unique common fixed point which is 2.

**Theorem 6.** Let $(\mathcal{M}, \ast)$ be a complete $\mathcal{M}$-FCM Space where $\mathcal{M}$ is triangular. If $\mathcal{P}, \Omega, \mathcal{R} : \mathcal{Z} \rightarrow \mathcal{Z}$ is such that for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in \text{int}(\mathcal{C})$,
\[
\left( \frac{1}{\mathcal{M}(\mathcal{P}\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c)} - 1 \right) \leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right),
\]
where $\Psi(\zeta, \eta, \omega) = \min\{\mathcal{M}(\zeta, \Omega\eta, \mathcal{R}\omega, c), \mathcal{M}(\mathcal{P}\zeta, \eta, \mathcal{R}\omega, c), \mathcal{M}(\mathcal{P}\zeta, \Omega\eta, \omega, c)\}$ and $k \in (0, 1)$. Then $\mathcal{P}, \Omega$ and $\mathcal{R}$ have unique common fixed point.
Proof. Let \( \zeta_0 \in \mathcal{Z} \) be arbitrary. Define the sequence \( \{ \zeta_n \} \) as in Theorem 2.

From (6.1),
\[
\frac{1}{M(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 = \frac{1}{M(P_{\zeta_{n+1}}, Q_{\zeta_{n+1}, \zeta_{n+1}}, c) - 1}
\leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right),
\]
where, \( \Psi(\zeta, \eta, \omega) = \min \left\{ \frac{M(\zeta_n, \zeta_{n+1}, Q_{\zeta_{n+1}}, c), M(P_{\zeta_{n+1}}, \zeta_{n+1}, c)}{M(P_{\zeta_{n+1}}, \zeta_{n+1}, c)} \right\} \)
\[
= \min \left\{ M(\zeta_n, \zeta_{n+2}, c), M(\zeta_{n+1}, \zeta_{n+2}, c) \right\}
= \min \left\{ M(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+2}, c) \right\}.
\]

Case (i) \( \Psi(\zeta, \eta, \omega) = M(\zeta_n, \zeta_{n+2}, c) \).
\[
\left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) \leq k \left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+2}, c)} - 1 \right)
\leq k \left( \frac{1}{M(\zeta_{n+2}, \zeta_{n+2}, \zeta_{n+1}, c)} - 1 \right)
\]
Therefore,
\[
\left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+2}, c)} - 1 \right) \leq k \left( \frac{1}{M(\zeta_n, \zeta_{n+1}, c)} - 1 \right).
\]

Case (ii) \( \Psi(\zeta, \eta, \omega) = M(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+2}, c) \).
\[
\left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) \leq k \left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right), \text{ and this gives}
\]
\[
\left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) = 0, \text{ which is absurd.}
\]
Therefore,
\[
\left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) \leq \frac{k}{1 - k} \left( \frac{1}{M(\zeta_n, \zeta_{n+1}, c)} - 1 \right). \tag{6.2}
\]

From (6.1),
\[
\left( \frac{1}{M(\zeta_{n+2}, \zeta_{n+3}, \zeta_{n+3}, c)} - 1 \right) = \frac{1}{M(Q_{\zeta_{n+2}, \zeta_{n+3}}, R_{\zeta_{n+2}, \zeta_{n+3}}, c)} - 1
\leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right),
\]
where, \( \Psi(\zeta, \eta, \omega) = \min \left\{ \frac{M(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}, c), M(Q_{\zeta_{n+2}, \zeta_{n+3}}, c)}{M(Q_{\zeta_{n+2}, \zeta_{n+3}}, c)} \right\} \)
\[
= \min \left\{ M(\zeta_{n+1}, \zeta_{n+3}, \zeta_{n+3}, c), M(\zeta_{n+2}, \zeta_{n+3}, c) \right\}
= \min \left\{ M(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+3}, c) \right\}.
\]

Case (i) \( \Psi(\zeta, \eta, \omega) = M(\zeta_{n+1}, \zeta_{n+3}, \zeta_{n+3}, c) \).
\[
\left( \frac{1}{M(\zeta_{n+2}, \zeta_{n+3}, \zeta_{n+3}, c)} - 1 \right) \leq k \left( \frac{1}{M(\zeta_{n+1}, \zeta_{n+3}, \zeta_{n+3}, c)} - 1 \right)
\leq k \left\{ \frac{1}{M(\zeta_{n+3}, \zeta_{n+3}, \zeta_{n+3}, c)} - 1 \right\}.
\]
Therefore,
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+4}, c)} - 1 \right) \leq \frac{k}{1 - k} \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c)} - 1 \right).
\]

**Case (ii)** \( \Psi(\zeta, \eta, \omega) = \mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+4}, c) \).

\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+4}, c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+2}, \zeta_{3n+3}, c)} - 1 \right), \text{ and, this gives}
\]
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+4}, c)} - 1 \right) = 0, \text{ which is absurd.}
\]
Therefore,
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+4}, c)} - 1 \right) \leq \frac{k}{1 - k} \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c)} - 1 \right). \tag{6.3}
\]
Again, from (6.1),
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right) = \left( \frac{1}{\mathcal{M}(R\zeta_{3n+2}, P\zeta_{3n+3}, P\zeta_{3n+3}, c)} - 1 \right)
\]
\[
\leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega) - 1} \right),
\]
where, \( \Psi(\zeta, \eta, \omega) = \min \left\{ \mathcal{M}(\zeta_{3n+2}, P\zeta_{3n+3}, P\zeta_{3n+3}, c), \mathcal{M}(R\zeta_{3n+2}, \zeta_{3n+3}, P\zeta_{3n+3}, c), \right\} \)
\[
= \min \left\{ \mathcal{M}(\zeta_{3n+2}, \zeta_{3n+4}, \zeta_{3n+4}, c), \mathcal{M}(\zeta_{3n+3}, \zeta_{3n+3}, \zeta_{3n+4}, c), \right\}
\]
\[
= \min \left\{ \mathcal{M}(\zeta_{3n+2}, \zeta_{3n+4}, \zeta_{3n+4}, c), \mathcal{M}(\zeta_{3n+3}, \zeta_{3n+3}, \zeta_{3n+4}, c) \right\}.
\]

**Case (i)** \( \Psi(\zeta, \eta, \omega) = \mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+4}, c) \).

\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right)
\]
\[
\leq k \left\{ \left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right) + \left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+2}, \zeta_{3n+2}, c)} - 1 \right) \right\}.
\]
Therefore,
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right) \leq \frac{k}{1 - k} \left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+2}, c)} - 1 \right).
\]

**Case (ii)** \( \Psi(\zeta, \eta, \omega) = \mathcal{M}(\zeta_{3n+3}, \zeta_{3n+3}, \zeta_{3n+4}, c) \).

\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right), \text{ and, this gives}
\]
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right) = 0, \text{ which is absurd.}
\]
Therefore,
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1 \right) \leq \frac{k}{1 - k} \left( \frac{1}{\mathcal{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+2}, c)} - 1 \right). \tag{6.4}
\]
From (6.2), (6.3) and (6.4), we obtain
\[
\left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) \leq \frac{k}{1 - k} \left( \frac{1}{\mathcal{M}(\zeta_{n}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1 \right), \text{ and, this gives,}
\]
\[
\left( \frac{1}{\mathcal{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1 \right) \leq \left( \frac{k}{1 - k} \right)^n \left( \frac{1}{\mathcal{M}(\zeta_{0}, \zeta_{1}, \zeta_{1}, c)} - 1 \right).
The above two inequalities imply that \( \{\zeta_n\} \) is \( \mathfrak{M}\)-FCC and Cauchy. Therefore there is an element \( \zeta \in \mathcal{Z} \) such that
\[
\lim_{n \to +\infty} \left( \frac{1}{\mathfrak{M}(\zeta_n, \zeta, \zeta, t)} - 1 \right) = 0.
\] (6.5)

Since \( \mathfrak{M} \) is triangular,
\[
\left( \frac{1}{\mathfrak{M}(\zeta, \zeta, P\zeta, t)} - 1 \right) \leq \left( \frac{1}{\mathfrak{M}(\zeta, \zeta, \zeta_3n+2, t)} - 1 \right) + \left( \frac{1}{\mathfrak{M}(\zeta_3n+2, P\zeta, P\zeta, t)} - 1 \right).
\] (6.6)

From (6.1),
\[
\left( \frac{1}{\mathfrak{M}(\zeta_3n+2, P\zeta, P\zeta, t)} - 1 \right) = \left( \frac{1}{\mathfrak{M}(\zeta_3n+1, P\zeta, P\zeta, t)} - 1 \right)
\leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right),
\] where, \( \Psi(\zeta, \eta, \omega) = \min \{ \mathfrak{M}(\zeta_3n+1, P\zeta, P\zeta, t), \mathfrak{M}(\zeta_3n+1, \zeta_3n+1, P\zeta, P\zeta, t) \} \)
\[
= \min \{ \mathfrak{M}(\zeta_3n+1, \zeta_3n+1, P\zeta, P\zeta, t), \mathfrak{M}(\zeta_3n+2, \zeta, P\zeta, t), \mathfrak{M}(\zeta_3n+2, P\zeta, P\zeta, t) \}
\to \mathfrak{M}(\zeta, \zeta, P\zeta, t) \quad \text{as} \quad n \to +\infty.
\]

Therefore,
\[
\lim_{n \to +\infty} \sup \left( \frac{1}{\mathfrak{M}(\zeta_n, P\zeta, P\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\mathfrak{M}(\zeta, P\zeta, P\zeta, t)} - 1 \right).
\] (6.7)

From (6.5), (6.6) and (6.7), we have that
\[
\left( \frac{1}{\mathfrak{M}(\zeta, P\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\mathfrak{M}(\zeta, P\zeta, t)} - 1 \right).
\] (6.8)

Therefore,
\[
\left( \frac{1}{\mathfrak{M}(\zeta, P\zeta, t)} - 1 \right) = 0
\]

This gives \( P\zeta = \zeta \). Since \( \mathfrak{M} \) is triangular,
\[
\left( \frac{1}{\mathfrak{M}(\zeta, \zeta, Q\zeta, t)} - 1 \right) \leq \left( \frac{1}{\mathfrak{M}(\zeta, \zeta, \zeta_3n+3, t)} - 1 \right) + \left( \frac{1}{\mathfrak{M}(\zeta_3n+3, Q\zeta, Q\zeta, t)} - 1 \right).
\] (6.9)

From (6.1),
\[
\left( \frac{1}{\mathfrak{M}(\zeta_3n+3, Q\zeta, Q\zeta, t)} - 1 \right) = \left( \frac{1}{\mathfrak{M}(Q\zeta_3n+2, Q\zeta, Q\zeta, t)} - 1 \right)
\leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right),
\] where, \( \Psi(\zeta, \eta, \omega) = \min \{ \mathfrak{M}(\zeta_3n+2, Q\zeta, Q\zeta, t), \mathfrak{M}(Q\zeta_3n+2, Q\zeta, Q\zeta, t) \} \)
\[
= \min \{ \mathfrak{M}(\zeta_3n+2, Q\zeta, Q\zeta, t), \mathfrak{M}(\zeta_3n+3, \zeta, Q\zeta, t), \mathfrak{M}(\zeta_3n+3, Q\zeta, Q\zeta, t) \}
\to \mathfrak{M}(\zeta, \zeta, Q\zeta, t) \quad \text{as} \quad n \to +\infty.
\]

Therefore,
\[
\lim_{n \to +\infty} \sup \left( \frac{1}{\mathfrak{M}(\zeta_n, Q\zeta, Q\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\mathfrak{M}(\zeta, Q\zeta, t)} - 1 \right).
\] (6.10)
From (6.5), (6.9) and (6.10), we have that
\[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, Q\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta, \zeta, Q\zeta, t)} - 1 \right).
\]
Therefore, \( \left( \frac{1}{\mathcal{M}(\zeta, \zeta, Q\zeta, t)} - 1 \right) = 0 \), and this gives,
\[
Q\zeta = \zeta. \quad (6.11)
\]
Since \( \mathcal{M} \) is triangular,
\[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\zeta, t)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(\zeta, \zeta, Q\zeta, t)} - 1 \right) + \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\zeta, R\zeta, t)} - 1 \right). \quad (6.12)
\]
From (6.1),
\[
\left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\zeta, R\zeta, t)} - 1 \right) = \left( \frac{1}{\mathcal{M}(p\zeta_{3n}, R\zeta, R\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right),
\]
where, \( \Psi(\zeta, \eta, \omega) = \min \left\{ \mathcal{M}(\zeta_{3n}, R\zeta, R\zeta, t), \mathcal{M}(p\zeta_{3n}, \zeta, R\zeta, t), \mathcal{M}(p\zeta_{3n}, R\zeta, \zeta, t) \right\} \)
\[
= \min \left\{ \mathcal{M}(\zeta_{3n}, R\zeta, R\zeta, t), \mathcal{M}(\zeta_{3n+1}, R\zeta, R\zeta, t), \mathcal{M}(\zeta_{3n+1}, \zeta, R\zeta, R\zeta, t) \right\} \to \mathcal{M}(\zeta, \zeta, R\zeta, t) \quad \text{as} \quad n \to +\infty.
\]
Therefore,
\[
\lim_{n \to +\infty} \sup \left( \frac{1}{\mathcal{M}(\zeta_{3n+1}, R\zeta, R\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\zeta, t)} - 1 \right). \quad (6.13)
\]
From (6.5), (6.12) and (6.13), we have
\[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\zeta, t)} - 1 \right).
\]
Therefore, \( \left( \frac{1}{\mathcal{M}(\zeta, \zeta, R\zeta, t)} - 1 \right) = 0 \), and this gives
\[
R\zeta = \zeta. \quad (6.14)
\]
From (6.8), (6.11) and (6.14), we get \( p\zeta = Q\zeta = R\zeta = \zeta \). Then from (6.1),
\[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, Q\zeta, t)} - 1 \right) = \left( \frac{1}{\mathcal{M}(p\zeta, Q\zeta, R\zeta, t)} - 1 \right) \leq k \left( \frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right),
\]
where, \( \Psi(\zeta, \eta, \omega) = \min \left\{ \mathcal{M}(\zeta, Q\zeta, R\zeta, t), \mathcal{M}(p\zeta, \zeta, R\zeta, t), \mathcal{M}(p\zeta, Q\zeta, \zeta, t) \right\} \)
\[
= \min \left\{ \mathcal{M}(\zeta, \zeta, R\zeta, t), \mathcal{M}(\zeta, \zeta, \zeta, t), \mathcal{M}(\zeta, \zeta, \zeta, t) \right\} \to \mathcal{M}(\zeta, \zeta, \zeta, t).
\]
Therefore,
\[
\left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, t)} - 1 \right) \leq k \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, t)} - 1 \right).
\]
Hence, \( \left( \frac{1}{\mathcal{M}(\zeta, \zeta, \zeta, t)} - 1 \right) = 0 \), and this gives,
\[
\zeta = \zeta.
\]
Hence we can conclude that \( p, Q \) and \( R \) have a unique common fixed point. \[\blacksquare\]
Example 7. Consider the $\mathcal{M}$-FCM Space given in Example (5) with $\mathcal{Z} = [0, +\infty]$ and the self mappings $P, Q$ and $R$ from $\mathcal{Z}$ to $\mathcal{Z}$, given by $P\zeta = \frac{2}{3}\zeta + 1$, $Q\eta = \frac{1}{3}\eta + 2$, and $R\omega = 3$. It is easily seen that condition (6.1) holds and therefore $P, Q$ and $R$ have a unique common fixed point and it is 3.

Corollary 8. Let $(\mathcal{Z}, \mathcal{M}, \ast)$ be a complete $\mathcal{M}$-FCM Space where $\mathcal{M}$ is triangular. If $P : \mathcal{Z} \to \mathcal{Z}$ is such that for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in \text{int}(\mathcal{C})$,

$$\left(\frac{1}{\mathcal{M}(P\zeta, P\eta, P\omega, c)} - 1\right) \leq k \left(\frac{1}{\Psi(\zeta, \eta, \omega)} - 1\right)$$

where $\Psi(\zeta, \eta, \omega) = \min\{\mathcal{M}(\zeta, \eta, \omega, c), \mathcal{M}(P\zeta, \eta, P\omega, c), \mathcal{M}(P\zeta, P\eta, z, c)\}$ and $k \in (0, 1)$. Then $P$ has a unique fixed point.

Conclusion:

We constructed $\mathcal{M}$-fuzzy cone Banach contraction theorem and theorems which assure the common fixed points for three self mappings under generalized fuzzy contractive conditions in $\mathcal{M}$-fuzzy cone metric spaces. This work can be either extended or generalized to various kinds of other spaces.

References