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Asymptotic stability in Caputo-Hadamard fractional dynamic equations

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Abstract

In this work, we investigate the asymptotic stability of the zero solution for Caputo-Hadamard fractional dynamic equations on a time scale. We will make use of the Krasnoselskii fixed point theorem in a weighted Banach space to show new stability results.

Keywords: Fixed points Fractional dynamic equations Asymptotic stability Time scales.
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1. Introduction

Fractional dynamic equations without and with delay arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of fractional dynamic equations have received the attention of many authors, see [1]-[27], [29]-[31] and the references therein.

Fractional dynamic equations involving Riemann-Liouville and Caputo Δ-fractional derivatives have been studied extensively by several researchers, see [2], [11], [30], [31]. However, the literature on Hadamard dynamic equations is not yet as enriched. The Δ-fractional derivative due to Hadamard differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard Δ-fractional derivative contains a logarithmic function of arbitrary exponent, see [21].

In [11], Belaid et al. investigated the following Caputo fractional dynamic equation

$$\begin{cases} {}_{\mathbb{T}}^{C}D_{0+}^{\alpha}x(t) = f(t, x(t)), \ t \in [0,\infty)_{\mathbb{T}}, \\ x(0) = x_0, \ x^{\triangle}(0) = x_1, \end{cases}$$

where ${}_{\mathbb{T}}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative on $\mathbb{T}$ of order $1 < \alpha < 2$. By employing the Krasnoselskii fixed point theorem, the asymptotic stability of the zero solution has been established.

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In this paper, we extend the results in [11] by proving the stability and asymptotic stability of the zero solution for the following Caputo-Hadamard fractional dynamic equation

$$\begin{cases} {}_{\mathbb{T}}^{CH}\mathfrak{D}_{1^+}^{\alpha} x(t) = f(t, x(t)), \ t \in [1,\infty)_{\mathbb{T}}, \\ x(1) = x_0, \ x^{\triangle}(1) = x_1, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is an unbounded above time scale with $1 \in \mathbb{T}$, ${}_{\mathbb{T}}^{CH}\mathfrak{D}_{1^+}^{\alpha}$ is the Caputo-Hadamard Δ-fractional derivative on $\mathbb{T}$ of order $1 < \alpha < 2$, $x_0, x_1 \in \mathbb{R}$, $f : [1,\infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a rd-continuous function with $f(t,0) \equiv 0$. To prove the stability and asymptotic stability of the trivial solution, we transform (1.1) into an equivalent integral equation and then use the Krasnoselskii fixed point theorem. The obtained integral equation is the sum of two mappings, one is a compact and the other is a contraction.

2. Preliminaries

In this section, We use $C_{rd}([1,\infty)_{\mathbb{T}})$ for a space of rd-continuous functions where $[1,\infty)_{\mathbb{T}}$ is an interval.

Definition 2.1 ([13]). *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.*

Definition 2.2 ([13]). *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

Definition 2.3 ([13]). *A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided that it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limits exist at left-dense points of $\mathbb{T}$. The set of all rd-continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$.*

Definition 2.4 ([13]). *Let $t \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be a function. Then Δ-derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ with the property that for each $\varepsilon > 0$ there exists a neighborhood U of t in $\mathbb{T}$ such that*

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s| \text{ for all } s \in U.$$

Definition 2.5 ([13]). *A function $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is called a Δ-antiderivative of a function $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ provided that F is rd-continuous on $[a,b]_{\mathbb{T}}$ and Δ-differentiable on $[a,b)_{\mathbb{T}}$ and $F^{\Delta}(t) = f(t)$ for all $t \in [a,b)_{\mathbb{T}}$. Then we define the Δ-integral from a to b of f by*

$$\int_a^b f(t)\,\Delta t = F(b) - F(a).$$

Remark 2.6. *All rd-continuous bounded functions on $[a,b)_{\mathbb{T}}$ are delta integrable from a to b.*

Definition 2.7 ([2, 21]). *Assume $\mathbb{T}$ is a time scale, $[1,b]_{\mathbb{T}} \subseteq \mathbb{T}$ and the function x is an integrable function on $[1,b]_{\mathbb{T}}$, then Hadamard Δ-fractional integral of x is defined by*

$${}_{\mathbb{T}}^{H}\mathfrak{I}_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{\Delta s}{s},$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2.8 ([2, 21]). *Let $x : \mathbb{T} \to \mathbb{R}$ be a function. The Caputo-Hadamard Δ-fractional derivative of x is defined by*

$${}_{\mathbb{T}}^{CH}\mathfrak{D}_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s x^{\Delta}\right)^n (s) \frac{\Delta s}{s},$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer of α.

Lemma 2.9 ([2, 21]). *Let $0<\alpha<1$. Suppose $x\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right)$ and $x^{\triangle}$ exists almost every where on any bounded interval of $[1,\infty)_{\mathbb{T}}$. Then,*

$$ {}_{\mathbb{T}}^{CH}\mathfrak{I}_{1+}^{\alpha}\ {}_{\mathbb{T}}^{CH}\mathfrak{D}_{1+}^{\alpha}x\left(t\right)=x\left(t\right)-x\left(1\right). $$

Remark 2.10. *From Definitions 2.7, 2.8 and Lemma 2.9, it is easy to see that*

(1) *Let $0<\alpha<1$. If x is rd-continuous on $[1,\infty)_{\mathbb{T}}$, then ${}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha}\ {}_{\mathbb{T}}^{CH}\mathfrak{I}_{1+}^{\alpha}x\left(t\right)=x\left(t\right)$ holds for all $t\in[1,\infty)_{\mathbb{T}}$.*

(2) *The Caputo-Hadamard Δ-fractional derivative of a constant is equal to zero.*

In our discussion, the following Banach space plays a fundamental role. Let $h:[0,\infty)\to[1,+\infty)$ be a strictly increasing continuous function with $h\left(0\right)=1$, $h\left(\log t\right)\to\infty$ as $t\to\infty$, $h\left(\log s\right)h\left(\log\frac{t}{s}\right)\leq h\left(\log t\right)$ for all $1\leq s\leq t<\infty$. Let

$$ E=\left\{x\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right):\sup_{t\in[1,\infty)_{\mathbb{T}}}\left|x\left(t\right)\right|/h\left(\log t\right)<\infty\right\}. $$

Then, E is a Banach space with the norm $\|x\|=\sup_{t\in[1,\infty)_{\mathbb{T}}}\frac{|x(t)|}{h(\log t)}$. For more properties of this Banach space, see [26]. Also, let

$$ \|\varphi\|_{t}=\max\left\{\left|\varphi\left(s\right)\right|:1\leq s\leq t\right\}, $$

for any $t\in[1,\infty)_{\mathbb{T}}$, $\varphi\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right)$ and for any $\varepsilon>0$ let $\Im\left(\varepsilon\right)=\left\{x\in E:\|x\|\leq\varepsilon\right\}$.

Lemma 2.11. *Let $r\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right)$. Then $x\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right)$ is a solution of the following Cauchy type problem*

$$ \begin{cases} {}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha}x\left(t\right)=r\left(t\right),\ t\in[1,\infty)_{\mathbb{T}},\ 1<\alpha<2,\\ x\left(1\right)=x_{0},\ x^{\triangle}\left(1\right)=x_{1}, \end{cases} \tag{2.1} $$

if and only if x is a solution of the following Cauchy type problem

$$ \begin{cases} x^{\triangle}\left(t\right)={}_{\mathbb{T}}^{H}\mathfrak{I}_{1+}^{\alpha-1}r\left(t\right)+x_{1},\\ x\left(1\right)=x_{0}. \end{cases} \tag{2.2} $$

Proof. Clearly, if $\psi\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right)$, then ${}_{\mathbb{T}}^{H}\mathfrak{I}_{1+}^{\gamma}\psi\left(1\right)=0$ with $0<\gamma<1$.

1) Let $x\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right)$ be a solution of (2.1). For any $t\in[1,\infty)_{\mathbb{T}}$, Definition 2.8 shows that

$$ {}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha}x\left(t\right)={}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha-1}x^{\triangle}\left(t\right)=r\left(t\right). $$

By using Lemma 2.9, we have

$$ x^{\triangle}\left(t\right)=x^{\triangle}\left(1\right)+{}_{\mathbb{T}}^{H}\mathfrak{I}_{1+}^{\alpha-1}r\left(t\right)={}_{\mathbb{T}}^{H}\mathfrak{I}_{1+}^{\alpha-1}r\left(t\right)+x_{1}, $$

which means that x is a solution of (2.2).

2) Let x be a solution of (2.2). For any $t\in[1,\infty)_{\mathbb{T}}$, by Remark 2.10, it is easy to show that

$$ {}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha}x\left(t\right)={}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha-1}x^{\triangle}\left(t\right)={}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha-1}\ {}_{\mathbb{T}}^{H}\mathfrak{I}_{1+}^{\alpha-1}r\left(t\right)+{}_{\mathbb{T}}^{H}\mathfrak{D}_{1+}^{\alpha-1}x_{1}=r\left(t\right). $$

In addition, note that $r\in C_{rd}\left([1,\infty)_{\mathbb{T}}\right)$, we have

$$ x^{\triangle}\left(1\right)={}_{\mathbb{T}}^{H}\mathfrak{I}_{1+}^{\alpha-1}r\left(1\right)+x_{1}=x_{1}. $$

□

Lemma 2.12. *Let $k \in (0, \infty)$. Then $x \in C_{rd}([1,\infty)_{\mathbb{T}})$ is a solution of (1.1) if and only if*

$$x(t) = x_0 e_{\ominus k}(t,1) + \frac{1 - e_{\ominus k}(t,1)}{k} x_1 + k \int_1^t e_{\ominus k}(t,s) x^{\sigma}(s) \Delta s$$
$$+ \frac{1}{\Gamma(\alpha-1)} \int_1^t \int_u^t e_{\ominus k}(t,s) \left(\log \frac{s}{u}\right)^{\alpha-2} \Delta s f(u, x(u)) \frac{\Delta u}{u}. \tag{2.3}$$

Proof. Let $x \in C_{rd}([1,\infty)_{\mathbb{T}})$ be a solution of (1.1). By Lemma 2.11, we get

$$\begin{cases} x^{\triangle}(t) = {}^{H}_{\mathbb{T}}\mathfrak{I}^{\alpha}_{1^+}\left(f(t, x(t))\right) + x_1, \ t \in [1,\infty)_{\mathbb{T}}, \\ x(1) = x_0, \ t \in [1,\infty)_{\mathbb{T}}. \end{cases}$$

Then

$$\begin{cases} x^{\triangle}(t) = \frac{1}{\Gamma(\alpha-1)} \int_1^t (\log \frac{t}{s})^{\alpha-2} f(s, x(s)) \frac{\Delta s}{s} + x_1, \\ x(1) = x_0, \ t \in [1,\infty)_{\mathbb{T}}. \end{cases} \tag{2.4}$$

So, we can write (2.4) as

$$\begin{cases} x^{\triangle}(t) + k x^{\sigma}(t) = k x^{\sigma}(t) + \frac{1}{\Gamma(\alpha-1)} \int_1^t (\log \frac{t}{s})^{\alpha-2} f(s, x(s)) \frac{\Delta s}{s} + x_1, \\ x(1) = x_0, \ t \in [1,\infty)_{\mathbb{T}}. \end{cases}$$

We obtain (2.3) by using the variation of constants formula. The converse follows easily because each step is reversible. This completes the proof. □

Definition 2.13. *The zero solution $x = 0$ of (1.1) is said to be*

(i) *stable in Banach space E, if for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $|x_0| + |x_1| \le \delta$ implies that the solution $x(t) = x(t, x_0, x_1)$ exists for all $t \in [1,\infty)_{\mathbb{T}}$ and satisfies $\|x\| \le \varepsilon$.*

(ii) *asymptotically stable, if it is stable in Banach space E and there is a number $\sigma > 0$ such that $|x_0| + |x_1| \le \sigma$ implies $\lim_{t\to\infty} \|x(t)\| = 0$.*

Lastly in this section, we state the Krasnoselskii fixed point theorem which enables us to prove the stability and asymptotic stability of the trivial solution to (1.1).

Theorem 2.14 (Krasnoselskii [28]). *Let Ω be a closed convex nonempty subset of a Banach space $(S, \|.\|)$. Assume that A and B map Ω into S such that*

(i) *$Ax + By \in \Omega$ for all $x, y \in \Omega$,*

(ii) *A is continuous and $A\Omega$ is contained in a compact set of S,*

(iii) *B is a contraction with constant $l < 1$.*

Then, there is a $x \in \Omega$ with $Ax + Bx = x$.

The following modified compactness criterion is needed in order to show (ii).

Theorem 2.15 ([26]). *Let $\mathcal{M}$ be a subset of the Banach space E. Then, $\mathcal{M}$ is relatively compact in E if the following conditions are satisfied*

(i) *$\{x(t)/h(\log t) : x \in \mathcal{M}\}$ is uniformly bounded,*

(ii) *$\{x(t)/h(\log t) : x \in \mathcal{M}\}$ is equicontinuous on any compact interval of $[1,\infty)_{\mathbb{T}}$,*

(iii) *$\{x(t)/h(\log t) : x \in \mathcal{M}\}$ is equiconvergent at infinity i.e. for any given $\varepsilon > 0$, there is a $T_0 > 1$ such that for all $x \in \mathcal{M}$ and $t_1, t_2 > T_0$, if holds*

$$|x(t_2)/h(\log t_2) - x(t_1)/h(\log t_1)| < \varepsilon.$$

3. Main results

We introduce the following hypotheses.

$(h1)$ There is a constant $\beta_1 \in (0,1)$ such that

$$\begin{aligned} & e_{\ominus k}(t,1)/h(\log t) \in BC([1,\infty)_{\mathbb{T}}) \cap L^1_{\triangle}([1,\infty)_{\mathbb{T}}), \\ & k\int_1^t e_{\ominus k}(t,1)/h(\log u)\Delta u \leq \beta_1 < 1. \end{aligned} \tag{3.1}$$

$(h2)$ There exist constants $\eta > 0$, $\beta_2 \in (0, 1-\beta_1)$ and a continuous function $\tilde{f} : [1,\infty)_{\mathbb{T}} \times (0,\eta] \to \mathbb{R}^+$ such that

$$\frac{|f(t, \upsilon h(\log t))|}{h(\log t)} \leq \tilde{f}(t, |\upsilon|), \tag{3.2}$$

holds for all $t \in [1,\infty)_{\mathbb{T}}$, $0 < |\upsilon| \leq \eta$ and

$$\sup_{t\in[1,\infty)_{\mathbb{T}}} \int_1^t \frac{K(\log \frac{t}{u})}{h(\log \frac{t}{u})} \frac{\tilde{f}(u,r)}{r} \frac{\Delta u}{u} \leq \beta_2 < 1-\beta_1, \tag{3.3}$$

holds for every $0 < r \leq \eta$, where $\tilde{f}(t,r)$ is nondecreasing in r for fixed t, $\tilde{f}(t,r) \in L^1([1,\infty)_{\mathbb{T}})$ in t for fixed r, and

$$K(\log \frac{t}{u}) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_u^t e_{\ominus k}(t,s)(\log \frac{s}{u})^{\alpha-2}\Delta s, & \log \frac{t}{u} \geq 0, \\ 0, & \log \frac{t}{u} < 0. \end{cases} \tag{3.4}$$

Theorem 3.1. *Assume that $(h1)$ and $(h2)$ hold. Then, the zero solution $x = 0$ of (1.1) is stable in Banach space E.*

Proof. First, for any given $\varepsilon > 0$, we show the existence of $\delta > 0$ such that

$$|x_0| + |x_1| < \delta \text{ implies } \|x\| \leq \varepsilon.$$

By (3.1), there is a constant $M_1 > 0$ such that

$$\frac{e_{\ominus k}(t,1)}{h(\log t)} \leq M_1. \tag{3.5}$$

Let $0 < \delta \leq \frac{(1-\beta_1-\beta_2)k}{M_1 k + 1 + M_1}\varepsilon$. Consider the closed convex nonempty subset $\Im(\varepsilon) \subseteq E$, for $t \in [1,\infty)_{\mathbb{T}}$, we denote two mapping A and B on $\Im(\varepsilon)$ as follows

$$\begin{aligned} (Ax)(t) &= \frac{1}{\Gamma(\alpha-1)} \int_1^t \int_u^t e_{\ominus k}(t,s)\left(\log \frac{s}{u}\right)^{\alpha-2} \Delta s f(u, x(u)) \frac{\Delta u}{u} \\ &= \int_1^t K\left(\log \frac{t}{u}\right) f(u,x(u)) \frac{\Delta u}{u}, \end{aligned} \tag{3.6}$$

and

$$(Bx)(t) = e_{\ominus k}(t,1)x_0 + \frac{1 - e_{\ominus k}(t,1)}{k}x_1 + k\int_1^t e_{\ominus k}(t,s)x^\sigma(s)\Delta s. \tag{3.7}$$

Obviously for $x \in \Im(\varepsilon)$, both Ax and Bx are rd-continuous functions on $[1,\infty)_{\mathbb{T}}$. In addition, for $x \in \Im(\varepsilon)$, by (3.1)-(3.3) for any $t \in [1,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} \frac{|(Ax)(t)|}{h(\log t)} &\leq \int_1^t \frac{K(\log \frac{t}{u})}{h(\log \frac{t}{u})} \frac{|f(u,x(u))|}{h(\log u)} \frac{\Delta u}{u} \\ &\leq \int_1^t \frac{K(\log \frac{t}{u})}{h(\log \frac{t}{u})} \tilde{f}\left(u, \frac{|x(u)|}{h(\log u)}\right) \frac{\Delta u}{u} \\ &\leq \beta_2 \|x\| \leq \beta_2\varepsilon < \infty, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\frac{|(Bx)(t)|}{h(\log t)} &= \left|\frac{e_{\ominus k}(t,1)}{h(\log t)}x_0 + \frac{1-e_{\ominus k}(t,1)}{kh(\log t)}x_1 + k\int_1^t \frac{e_{\ominus k}(t,s)}{h(\log t)}x^\sigma(s)\Delta s\right| \\ &\le M_1|x_0| + \frac{1+M_1}{k}|x_1| + k\int_1^\infty \frac{e_{\ominus k}(u,1)}{h(\log u)}\Delta u\,\|x\| \\ &\le M_1|x_0| + \frac{1+M_1}{k}|x_1| + \beta_1\varepsilon < \infty. \end{aligned} \tag{3.9}$$

Then $A\Im(\varepsilon) \subseteq E$ and $B\Im(\varepsilon) \subseteq E$. Next, we shall use Theorem 2.14 to show there is a fixed point of the operator $A+B$ in $\Im(\varepsilon)$. Here, we divide the proof into three steps.

Step1. We show that $Ax + By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$.
For any $x, y \in \Im(\varepsilon)$, by (3.8) and (3.9), we get that

$$\begin{aligned}&\sup_{t\in[1,\infty)_{\mathbb{T}}} \frac{|(Ax)(t)+(By)(t)|}{h(\log t)} \\ &= \sup_{t\in[1,\infty)_{\mathbb{T}}}\left\{\left|\frac{e_{\ominus k}(t,1)}{h(\log t)}x_0 + \frac{1-e_{\ominus k}(t,1)}{kh(\log t)}x_1 + k\int_1^t \frac{e_{\ominus k}(t,s)}{h(\log t)}y^\sigma(s)\Delta s\right.\right. \\ &\left.\left. + \int_1^t \frac{K(\log\frac{t}{u})}{h(\log t)}f(u,x(u))\frac{\Delta u}{u}\right|\right\} \\ &\le M_1|x_0| + \frac{1+M_1}{k}|x_1| + k\int_1^\infty \frac{e_{\ominus k}(t,1)}{h(\log u)}\Delta u\,\|y\| + \beta_2\|x\| \\ &\le \frac{M_1k+1+M_1}{k}\delta + \beta_1\varepsilon + \beta_2\varepsilon \le \varepsilon, \end{aligned}$$

which means that $Ax + By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$.

Step 2. It is easy to prove that A is continuous. Now, we only show that $A\Im(\varepsilon)$ is a relatively compact in E. By (3.8), we obtain that $\{x(t)/h(\log t) : x \in \Im(\varepsilon)\}$ is uniformly bounded in E. Also, a classical theorem states the fact that the convolution of an $L^1_\triangle$-function with a function tending to zero, does also tend to zero. Then we conclude that for $\left(\log\frac{t}{u}\right) \ge 0$, we get

$$\begin{aligned} 0 \le \lim_{t\to\infty}\frac{K(\log\frac{t}{u})}{h(\log\frac{t}{u})} &\le \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_u^t \frac{e_{\ominus k}(t,s)}{h(\log\frac{t}{s})}\frac{(\log\frac{s}{u})^{\alpha-2}}{h(\log\frac{s}{u})}\Delta s \\ &\le \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_1^t \frac{e_{\ominus k}(t,us)}{h(\log\frac{t}{us})}\frac{(\log s)^{\alpha-2}}{h(\log s)}u\Delta s = 0, \end{aligned} \tag{3.10}$$

because $\lim_{t\to\infty}\frac{(\log t)^{\alpha-2}}{h(\log t)} = 0$. Together with the continuity of K and h, we obtain that there is a constant $M_2 > 0$ such that

$$\left|\frac{K\left(\log\frac{t}{u}\right)}{h(\log\frac{t}{u})}\right| \le M_2, \tag{3.11}$$

and for any $T_0 \in [1,\infty)_{\mathbb{T}}$, the function $K(\log\frac{t}{u})h(\log u)/h(\log t)$ is uniformly continuous on $\{(t,u) : 1 \le u \le$

$t \leq T_0\}$. For any $t_1, t_2 \in [1, T_0] \cap \mathbb{T}$, $t_1 < t_2$, we get

$$\begin{aligned}
&\left| \frac{(Ax)(t_2)}{h(\log t_2)} - \frac{(Ax)(t_1)}{h(\log t_1)} \right| \\
&= \left| \int_1^{t_2} \frac{K(\log \frac{t_2}{u})}{h(\log t_2)} f(u, x(u)) \frac{\Delta u}{u} - \int_1^{t_1} \frac{K(\log \frac{t_1}{u})}{h(\log t_1)} f(u, x(u)) \frac{\Delta u}{u} \right| \\
&\leq \int_1^{t_1} \left| \frac{K(\log \frac{t_2}{u})}{h(\log t_2)} - \frac{K(\log \frac{t_1}{u})}{h(\log t_1)} \right| |f(u, x(u))| \frac{\Delta u}{u} \\
&+ \int_{t_1}^{t_2} \frac{K(\log \frac{t_2}{u})}{h(\log \frac{t_2}{u})} \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \\
&\leq \int_1^{t_1} \left| \frac{K(\log \frac{t_2}{u}) h(\log u)}{h(\log t_2)} - \frac{K(\log \frac{t_1}{u}) h(\log u)}{h(\log t_1)} \right| \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \\
&+ M_2 \int_{t_1}^{t_2} \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \to 0,
\end{aligned}$$

as $t_2 \to t_1$, which implies that $\{x(t)/h(\log t) : x \in \Im(\varepsilon)\}$ is equicontinuous on any compact interval of $[1, \infty)_{\mathbb{T}}$. By using Theorem 2.15, in order to prove that $A\Im(\varepsilon)$ is a relatively compact set of E, we only need to show that $\{x(t)/h(\log t) : x \in \Im(\varepsilon)\}$ is equiconvergent at infinity. In fact, for any $\varepsilon_1 > 0$, there is a $L > 1$ such that

$$M_2 \int_L^{\infty} \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \leq \frac{\varepsilon_1}{3}.$$

By (3.10), we obtain that

$$\lim_{t \to \infty} \sup_{u \in [1, L]_{\mathbb{T}}} \frac{K(\log \frac{t}{u})}{h(\log \frac{t}{u})} \leq \max \left\{ \lim_{t \to \infty} \frac{K\left(\log \frac{t}{L}\right)}{h\left(\log \frac{t}{L}\right)}, \lim_{t \to \infty} \frac{K(\log t)}{h(\log t)} \right\} = 0.$$

So, there is $T > L$ such that $t_1, t_2 \geq T$, we get

$$\begin{aligned}
&\sup_{u \in [1, L]_{\mathbb{T}}} \left| \frac{K(\log \frac{t_2}{u}) h(\log u)}{h(\log t_2)} - \frac{K(\log \frac{t_1}{u}) h(\log u)}{h(\log t_1)} \right| \\
&\leq \sup_{u \in [1, L]_{\mathbb{T}}} \left| \frac{K(\log \frac{t_2}{u})}{h(\log \frac{t_2}{u})} \right| + \sup_{u \in [1, L]_{\mathbb{T}}} \left| \frac{K(\log \frac{t_1}{u})}{h(\log \frac{t_1}{u})} \right| \\
&\leq \frac{\varepsilon_1}{3} \left(\int_1^{\infty} \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \right)^{-1}.
\end{aligned}$$

Thus, for $t_1, t_2 \geq T$,

$$\begin{aligned}
&\left| \frac{(Ax)(t_2)}{h(\log t_2)} - \frac{(Ax)(t_1)}{h(\log t_1)} \right| \\
&= \left| \int_1^{t_2} \frac{K(\log \frac{t_2}{u})}{h(\log t_2)} f(u, x(u)) \frac{\Delta u}{u} - \int_1^{t_1} \frac{K(\log \frac{t_1}{u})}{h(\log t_1)} f(u, x(u)) \frac{\Delta u}{u} \right| \\
&\leq \int_1^{L} \left| \frac{K(\log \frac{t_2}{u}) h(\log u)}{h(\log t_2)} - \frac{K(\log \frac{t_1}{u}) h(\log u)}{h(\log t_1)} \right| \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \\
&+ \int_L^{t_2} \frac{K(\log \frac{t_2}{u})}{h(\log \frac{t_2}{u})} \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} + \int_L^{t_1} \frac{K(\log \frac{t_1}{u})}{h(\log \frac{t_1}{u})} \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \\
&\leq \frac{\varepsilon_1}{3} + 2M_2 \int_L^{\infty} \tilde{f}(u, \varepsilon) \frac{\Delta u}{u} \leq \varepsilon_1.
\end{aligned}$$

Hence, the required conclusion is true.

Step 3. We prove that $B : \Im(\varepsilon) \to E$ is a contraction mapping.
For any $x, y \in \Im(\varepsilon)$, by (3.1), we get that

$$\begin{aligned}
&\sup_{t\in[1,\infty)_{\mathbb{T}}} \left| \frac{(Bx)(t)}{h(\log t)} - \frac{(By)(t)}{h(\log t)} \right| \\
&= \sup_{t\in[1,\infty)_{\mathbb{T}}} \left\{ \left| \frac{k\int_1^t e_{\ominus k}(t,u)\, x^\sigma(u)\Delta u}{h(\log t)} - \frac{k\int_1^t e_{\ominus k}(t,u)\, y^\sigma(u)\Delta u}{h(\log t)} \right| \right\} \\
&\le \sup_{t\in[1,\infty)_{\mathbb{T}}} k \int_1^t \frac{e_{\ominus k}(t,u)}{h(\log \frac{t}{u})} \frac{|x^\sigma(u) - y^\sigma(u)|}{h(\log u)} \Delta u \\
&\le k \int_1^t \frac{e_{\ominus k}(t,u)}{h(\log \frac{t}{u})} \Delta u \, \|x - y\| \\
&\le \beta_1 \|x - y\| .
\end{aligned}$$

By using Theorem 2.14, we know that there is a fixed point of the operator $A + B$ in $\Im(\varepsilon)$. Finally, for any $\varepsilon_2 > 0$, if $0 < \delta_1 \le \frac{(1-\beta_1-\beta_2)k}{kM_1+1+M_1}\varepsilon_2$, then $|x_0| + |x_1| \le \delta_1$ means that

$$\begin{aligned}
\|x\| = &\sup_{t\in[1,\infty)_{\mathbb{T}}} \left\{ \left| \frac{e_{\ominus k}(t,1)}{h(\log t)} x_0 + \frac{1 - e_{\ominus k}(t,1)}{kh(\log t)} x_1 + k \int_1^t \frac{e_{\ominus k}(t,s)}{h(\log t)} x^\sigma(u)\Delta u \right.\right. \\
&\left.\left. + \int_1^t \frac{K(\log \frac{t}{u})}{h(\log t)} f(u, x(u)) \frac{\Delta u}{u} \right| \right\} \\
\le &\sup_{t\in[1,\infty)_{\mathbb{T}}} \left\{ \frac{e_{\ominus k}(t,1)}{h(\log t)} |x_0| + \frac{|1 - e_{\ominus k}(t,1)|}{kh(\log t)} |x_1| + k \int_1^t \frac{e_{\ominus k}(t,u)\,|x^\sigma(u)|}{h(\log \frac{t}{u}) h(\log u)} \Delta u \right. \\
&\left. + \int_1^t \frac{K(\log \frac{t}{u})}{h(\log \frac{t}{u})} \frac{|f(u, x(u))|}{h(\log u)} \frac{\Delta u}{u} \right\} \\
\le &M_1\delta_1 + \frac{1 + M_1}{k}\delta_1 + \beta_1 \|x\| + \beta_2 \|x\| \\
\le &\frac{kM_1 + 1 + M_1}{(1 - \beta_1 - \beta_2)\, k} \delta_1 \le \varepsilon_2.
\end{aligned}$$

Then, the zero solution of (1.1) is stable in Banach space E. □

Theorem 3.2. *Assume that all conditions of Theorem 3.1 hold,*

$$\lim_{t\to\infty} e_{\ominus k}(t,1) / h(\log t) = 0, \tag{3.12}$$

and for any $r > 0$, there is a function $\varphi_r \in L^1_\triangle([1,\infty)_{\mathbb{T}})$, $\varphi_r(t) > 0$ such that $|u| \le r$ means

$$|f(t,u)| / h(\log t) \le \varphi_r(t), \ \text{a.e. } t \in [1,\infty)_{\mathbb{T}} . \tag{3.13}$$

Then, the zero solution of (1.1) is asymptotically stable.

Proof. It follows by Theorem 3.1 that the zero solution of (1.1) is stable in the Banach space E. Next, we shall prove that the zero solution $x = 0$ of (1.1) is attractive. For any $r > 0$, we define

$$\Im_*(r) = \left\{ x \in \Im(r), \ \lim_{t\to\infty} x(t)/h(\log t) = 0 \right\}.$$

We only need to show that $Ax + By \in \Im_*(r)$ for any $x, y \in \Im_*(r)$, i.e.

$$\frac{(Ax)(t) + (By)(t)}{h(\log t)} \to 0 \text{ as } t \to \infty,$$

where

$$\begin{aligned}&(Ax)(t)+(By)(t)\\&=e_{\ominus k}(t,1)\,x_0+\frac{1-e_{\ominus k}(t,1)}{k}x_1+k\int_1^t e_{\ominus k}(t,s)\,y^{\sigma}(u)\Delta u\\&+\int_1^t K\left(\log\frac{t}{u}\right)f(u,x(u))\frac{\Delta u}{u}.\end{aligned}$$

For $x,y\in\Im_*(r)$, based on the fact that used in the proof of Theorem 3.1 (Step2), it follows by using (3.1) and (3.12) that

$$\int_1^t \frac{e_{\ominus k}(t,u)}{h(\log\frac{t}{u})}\frac{y^{\sigma}(u)}{h(\log u)}\Delta u\to 0,$$

and

$$\frac{K(\log\frac{t}{u})}{h(\log\frac{t}{u})}=\frac{\int_u^t\frac{e_{\ominus k}(t,u)}{h(\log\frac{t}{u})}(\log\frac{s}{u})^{\alpha-2}\Delta s}{\Gamma(\alpha-1)}\to 0,$$

as $t\to\infty$. Together with the hypothesis $\varphi_r(t)\in L^1_{\triangle}([1,\infty)_{\mathbb{T}})$, we get that

$$\int_1^t\frac{K(\log\frac{t}{u})}{h(\log\frac{t}{u})}\frac{|f(u,x(u))|}{h(\log u)}\frac{\Delta u}{u}\le\int_1^t\frac{K(\log\frac{t}{u})}{h(\log\frac{t}{u})}\varphi_r(u)\frac{\Delta u}{u}\to 0,$$

as $t\to\infty$. Thus, we obtain the conclusion. □

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